



Misspecified Gaussian Process Bandits

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NeurIPS, 2021

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GP (kernel) bandits

- Problem setup:
 - ▶ Unknown reward function $f^\star : \mathcal{D} \rightarrow \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^d$ over a known infinite and compact set of actions
 - ▶ Interaction over T rounds: Select action $x_t \in \mathcal{D}$ and obtain noisy (sub-Gaussian) observation $y_t = f^\star(x_t) + \eta_t$
- Goal: Minimize cumulative regret $R_T = \sum_{t=1}^T \left(\max_{x \in D} f^\star(x) - f^\star(x_t) \right)$ (optimizing while learning)

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- Hypothesis class:
 - ▶ Defined via positive semi-definite kernel $k(\cdot, \cdot)$ (e.g., squared exponential, Matern, NTK)
 - ▶ Smooth functions with bounded RKHS norm $\mathcal{F}_k(\mathcal{D}; B) = \{f \in \mathcal{H}_k(\mathcal{D}) : \|f\|_k \leq B\}$
- Realizable setting: Assumed hypothesis class is the correct one, i.e., $f^\star \in \mathcal{F}_k(\mathcal{D}; B)$

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- Applications: Molecular and material design, AutoML, recommender systems and advertising, sensor nets, etc.

Misspecified GP (kernel) bandits

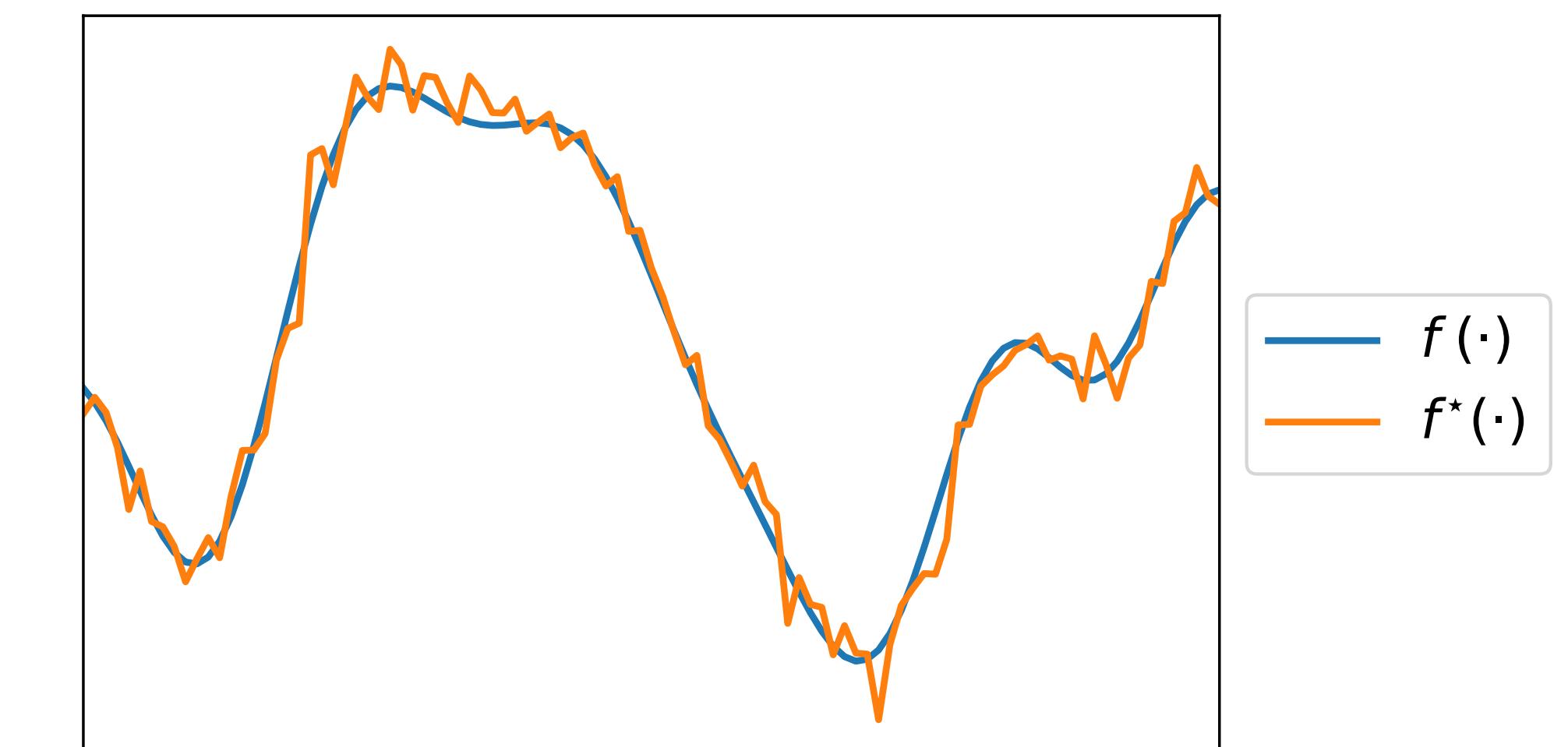
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- Assumption: The reward function can be uniformly approximated by a member from $\mathcal{F}_k(\mathcal{D}; B)$

$$\min_{f \in \mathcal{F}_k(\mathcal{D}; B)} \| f - f^* \|_\infty \leq \epsilon$$

- ▶ Misspecification rate $\epsilon > 0$ is unknown to the learner
- ▶ Misspecification rate $\epsilon = 0$ recovers the realizable setting



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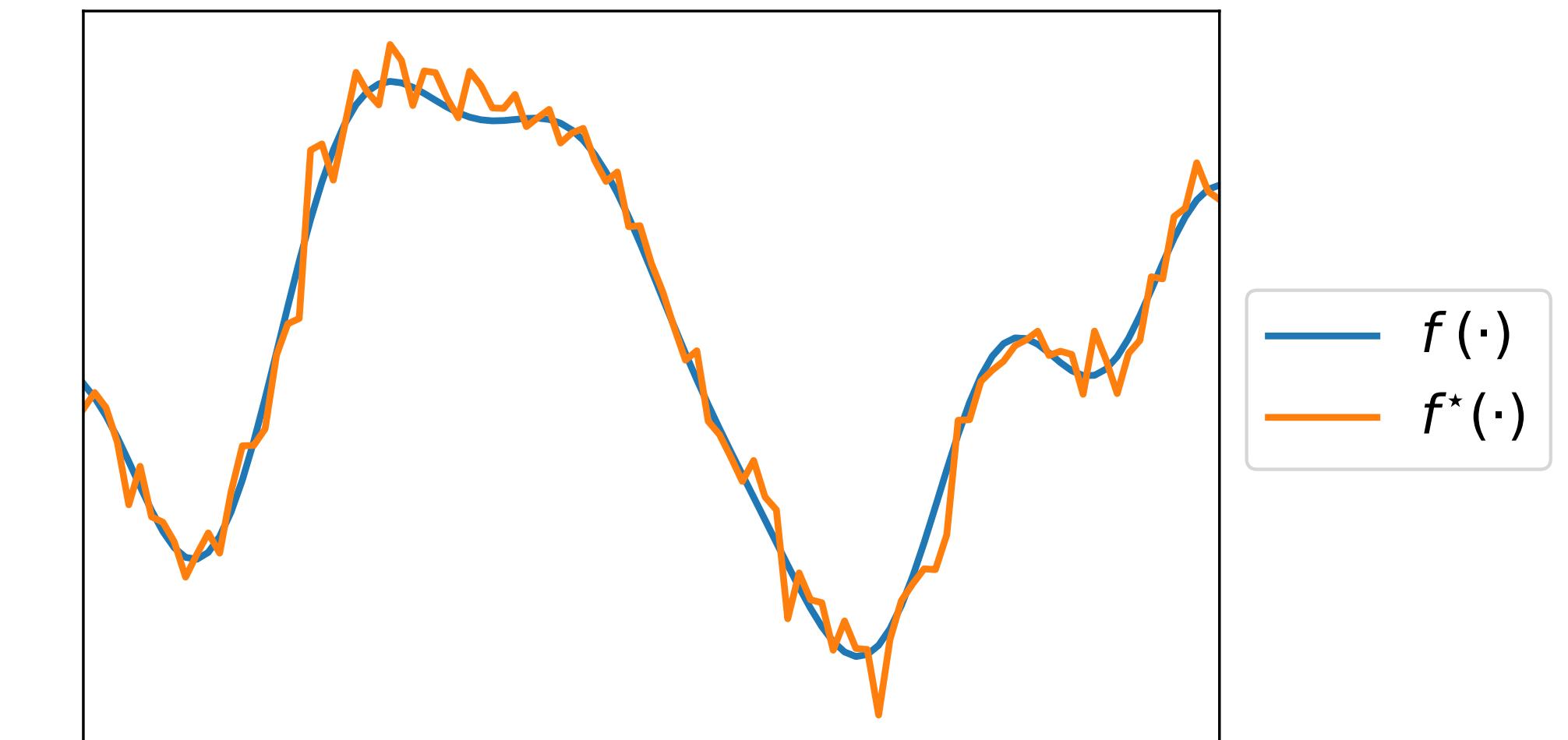
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Note: If $\|f - f'\|_k$ is small, then $\|f - f'\|_\infty$ is also small.

But the reverse does not need to hold !!

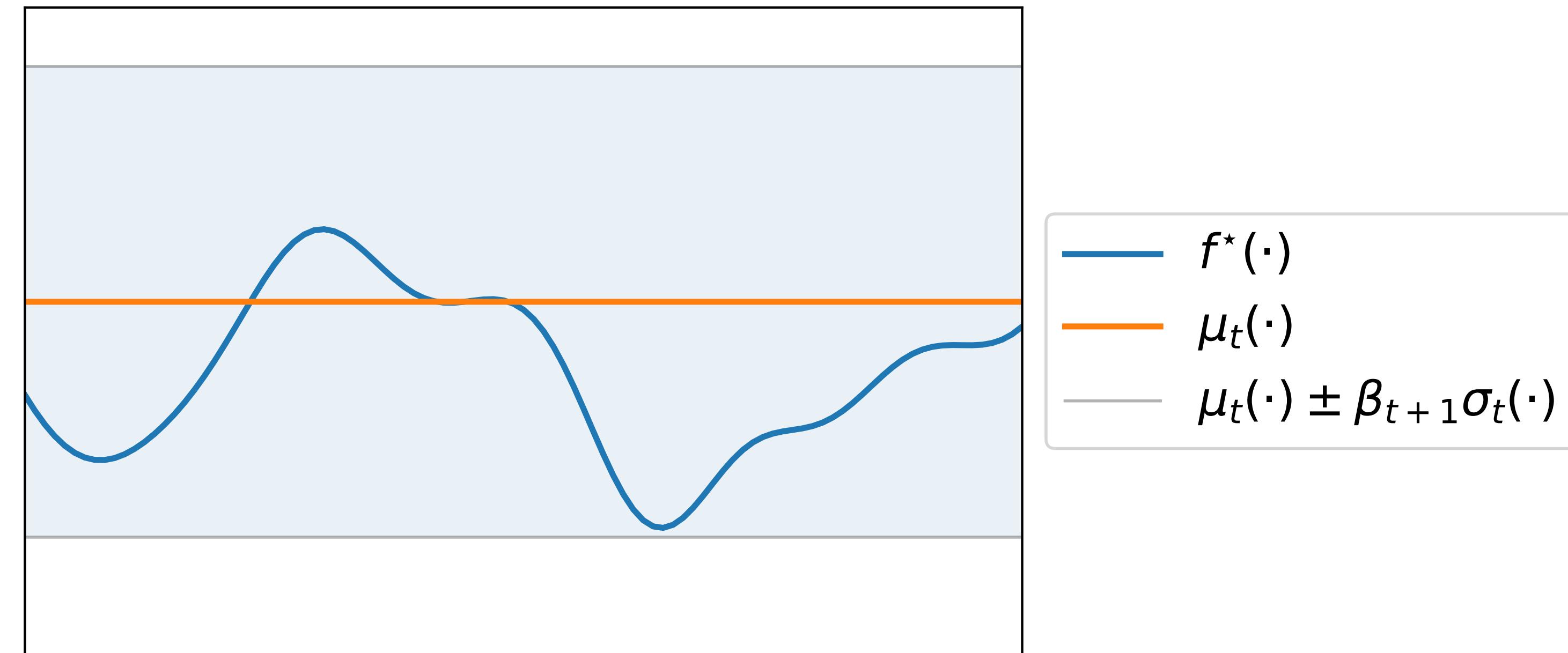


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- Kernel ridge regression / Gaussian Process posterior to obtain function estimate $\mu_t(x)$
- Key idea: Use estimate $\mu_t(x)$ to guide exploration

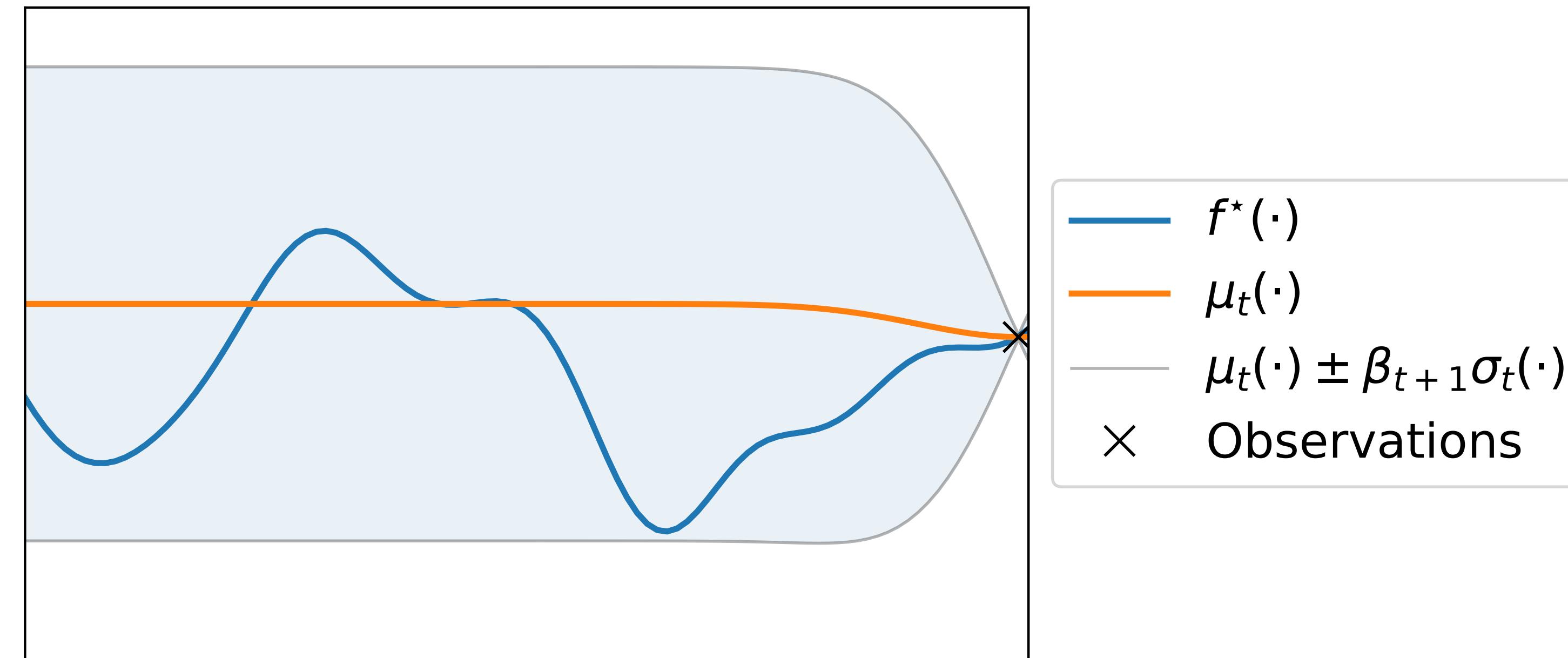
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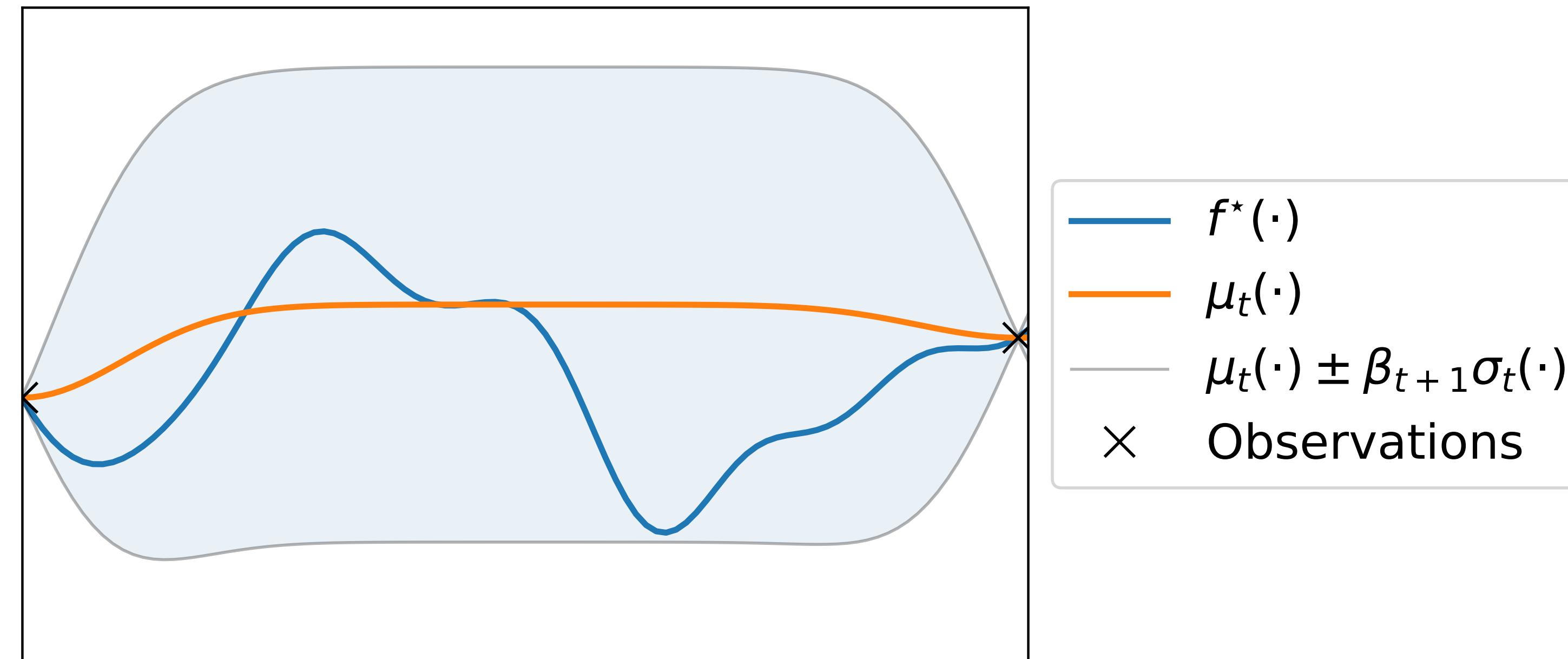
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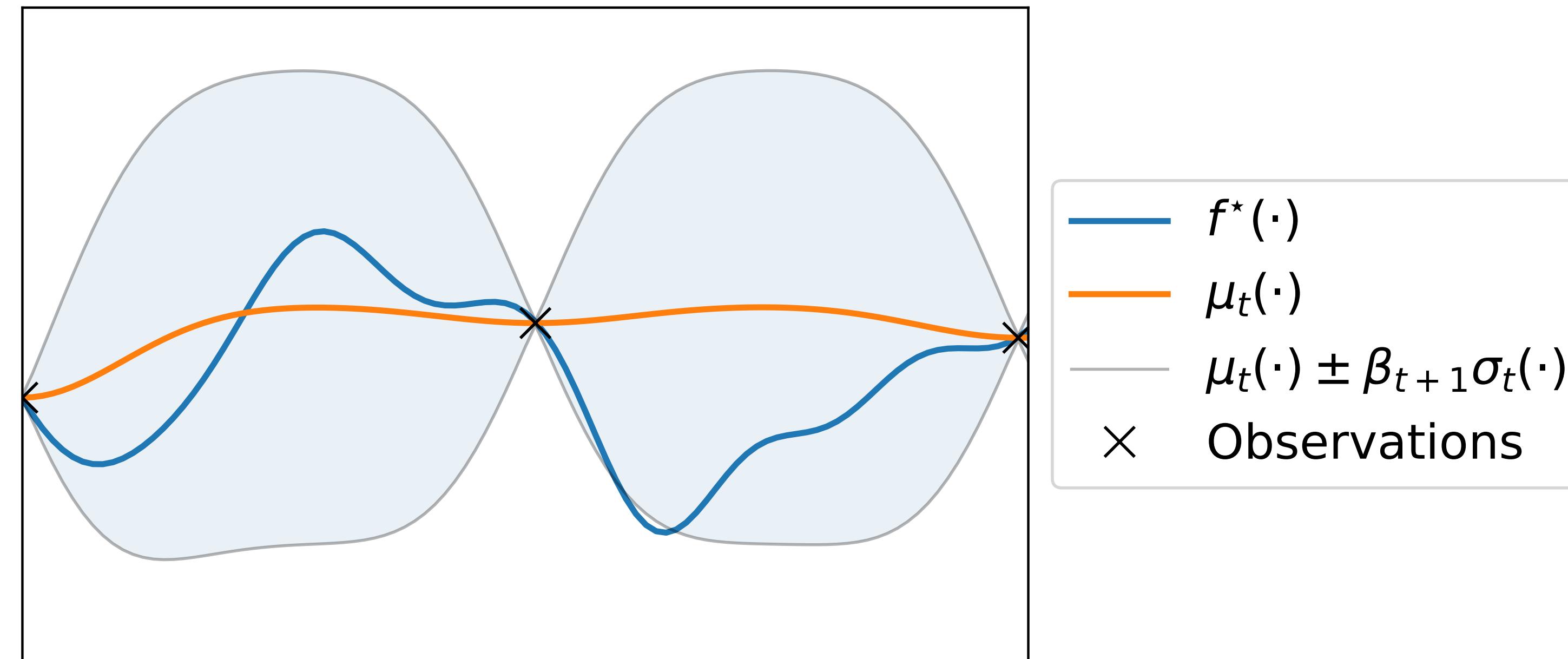
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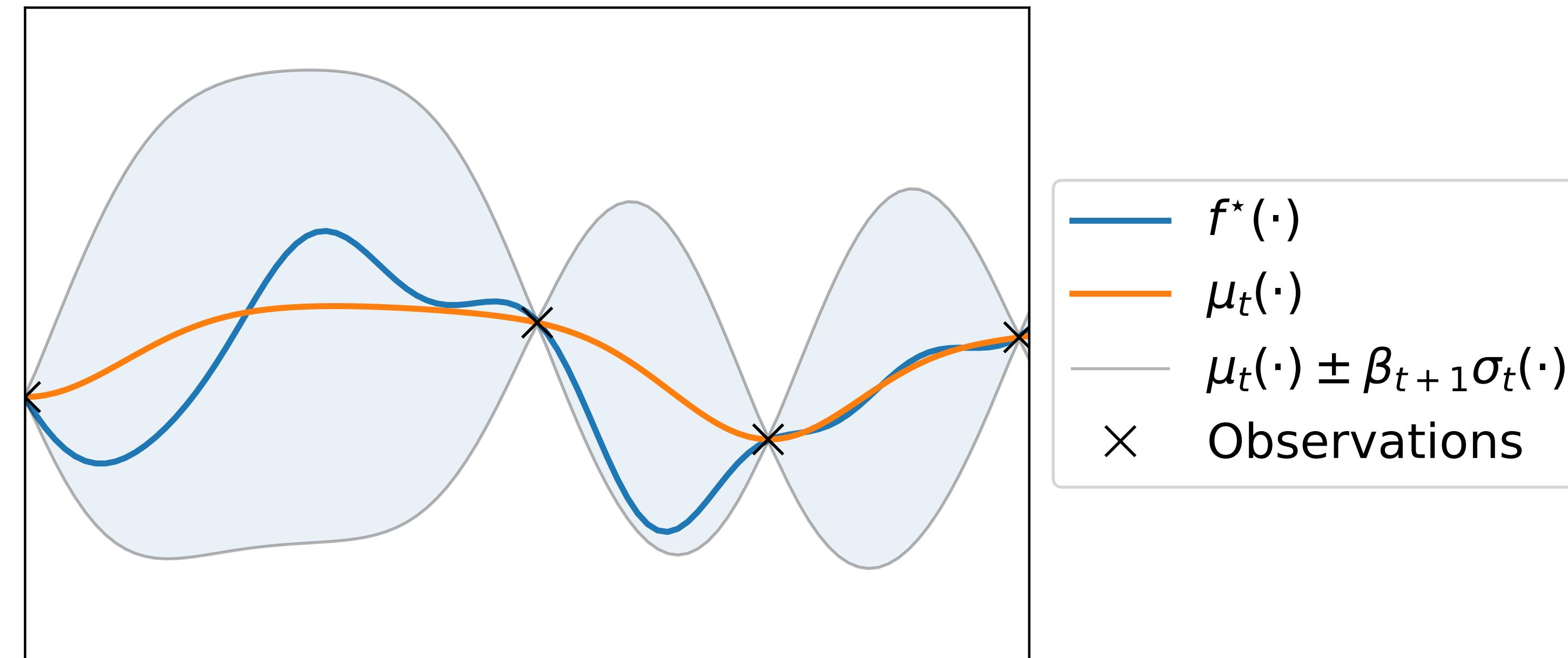
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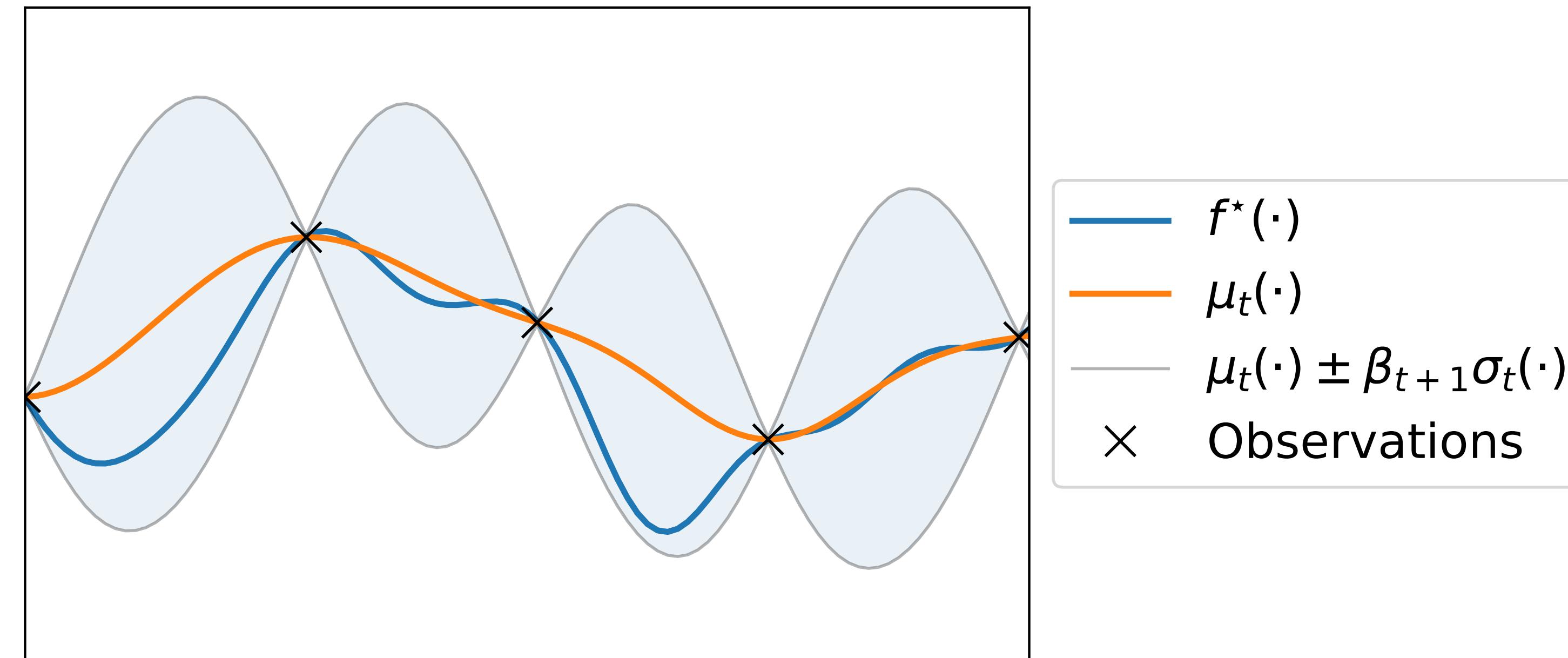
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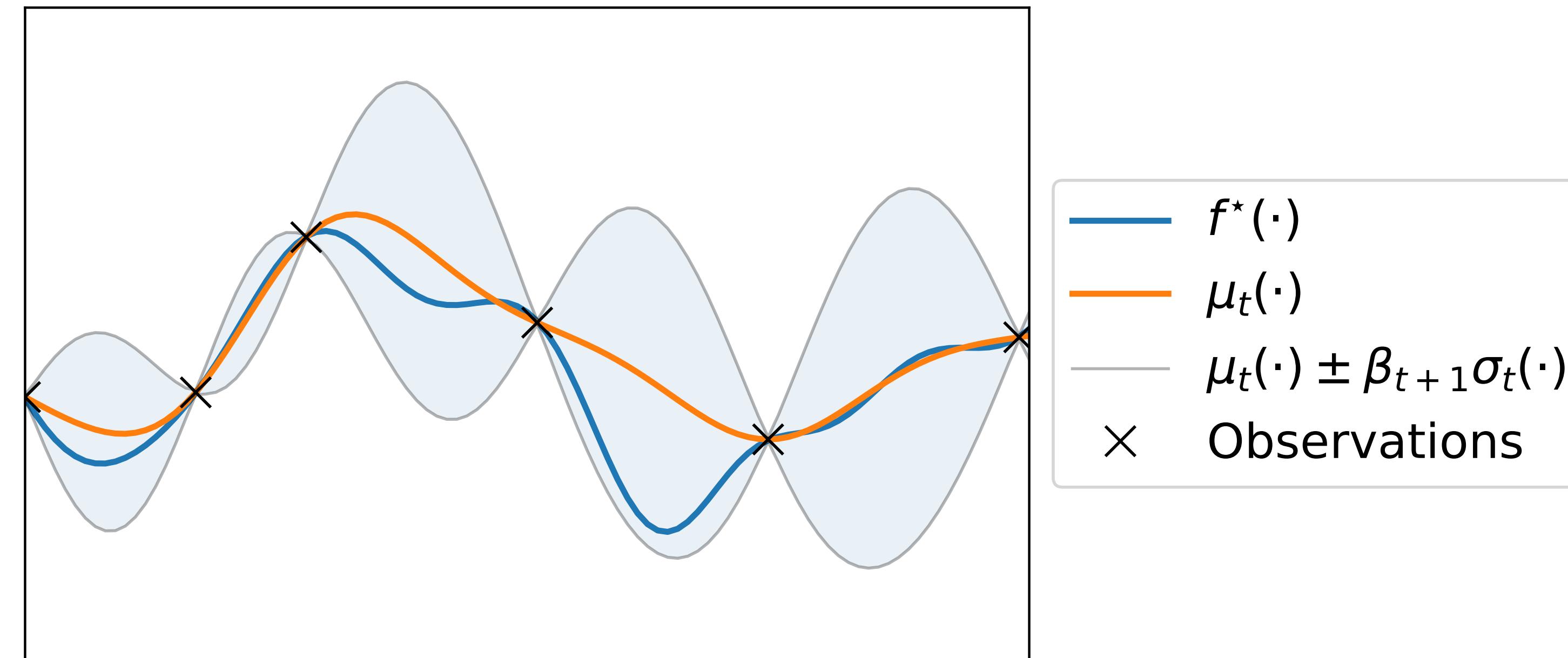
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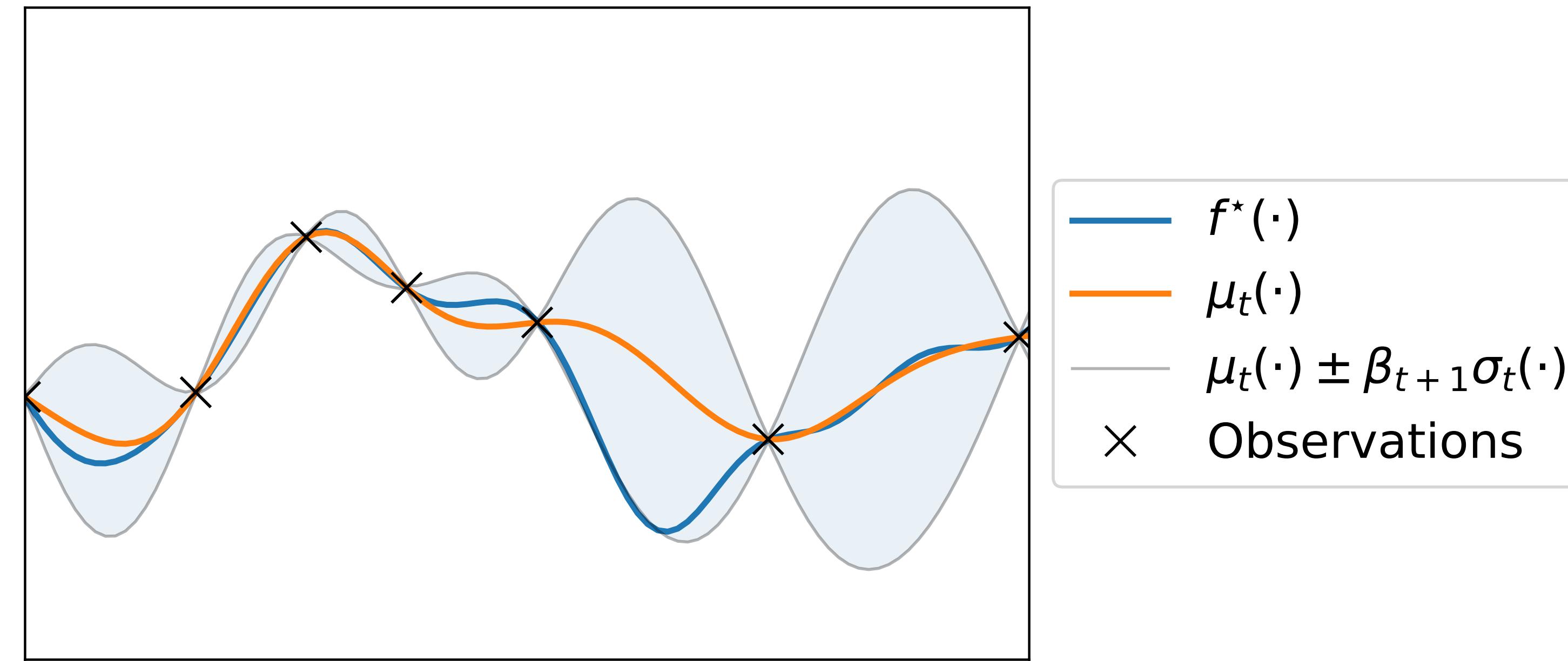
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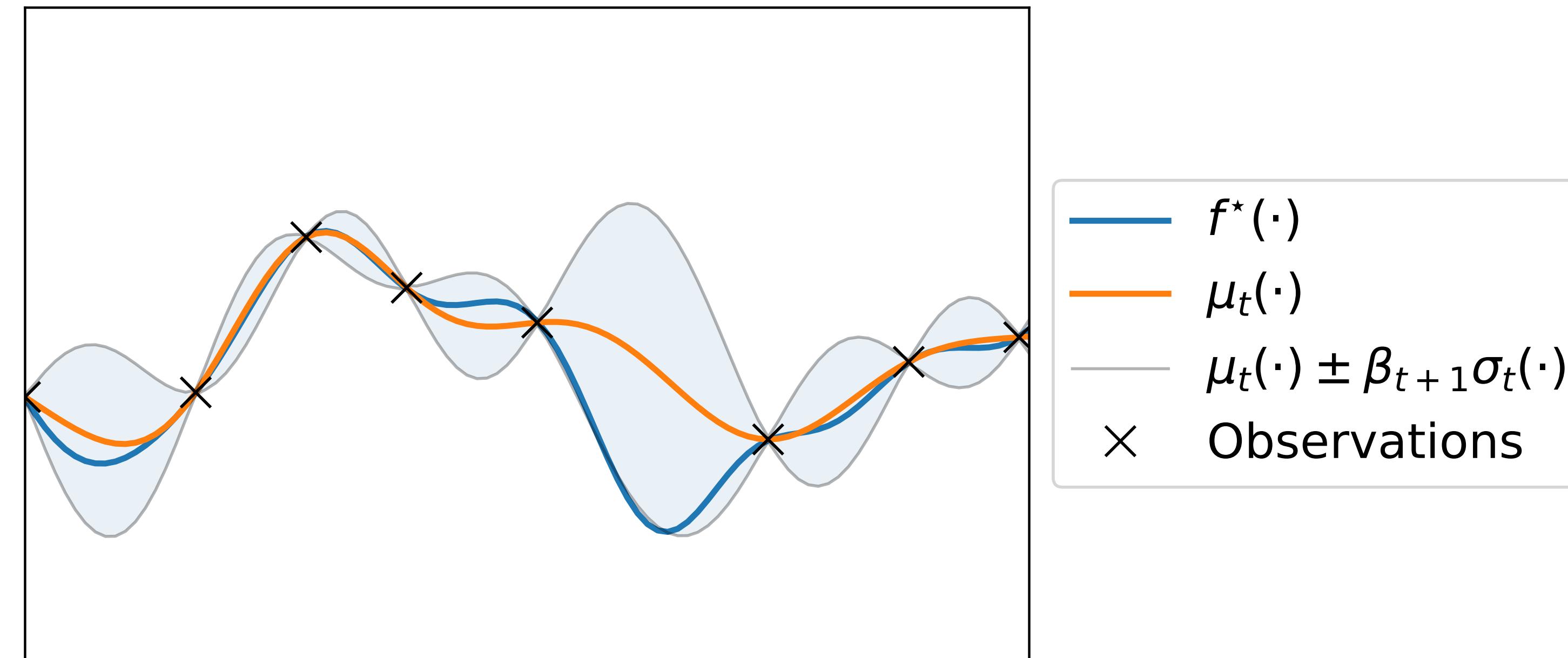
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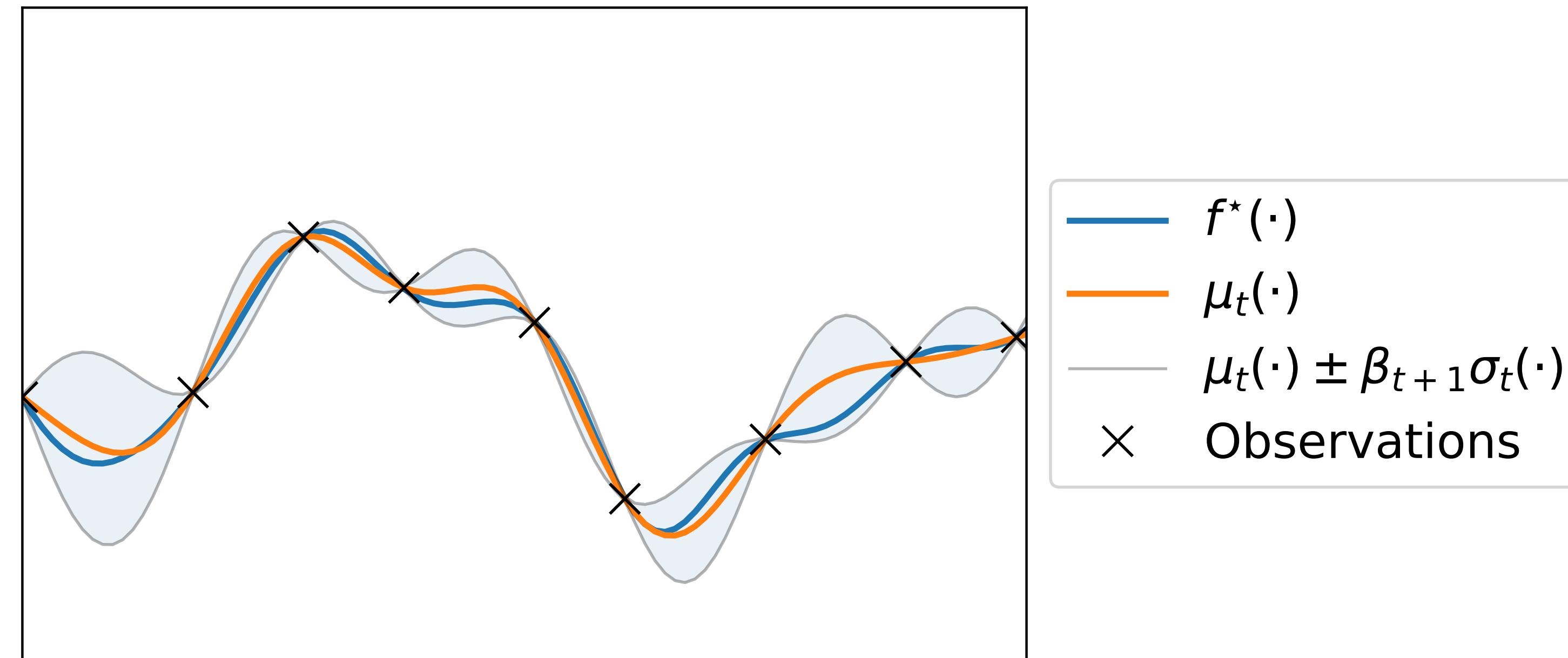
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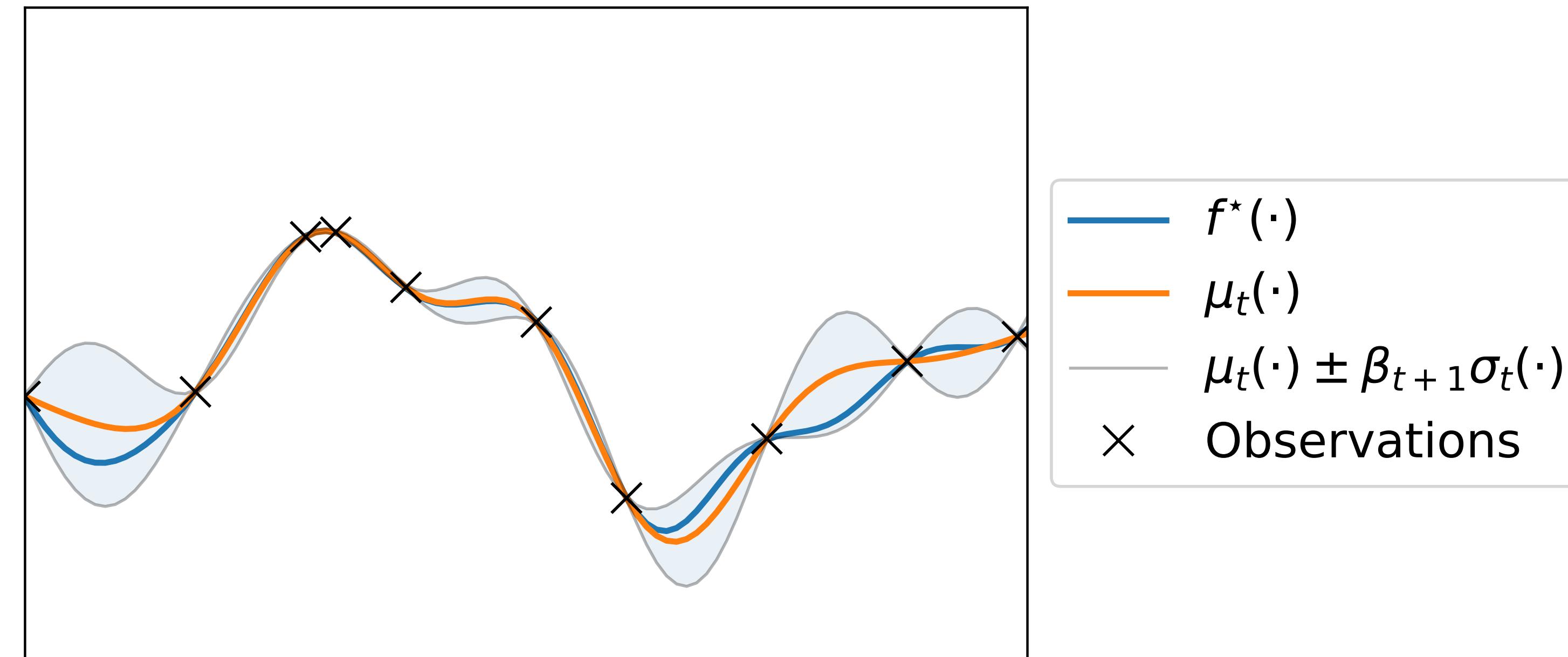
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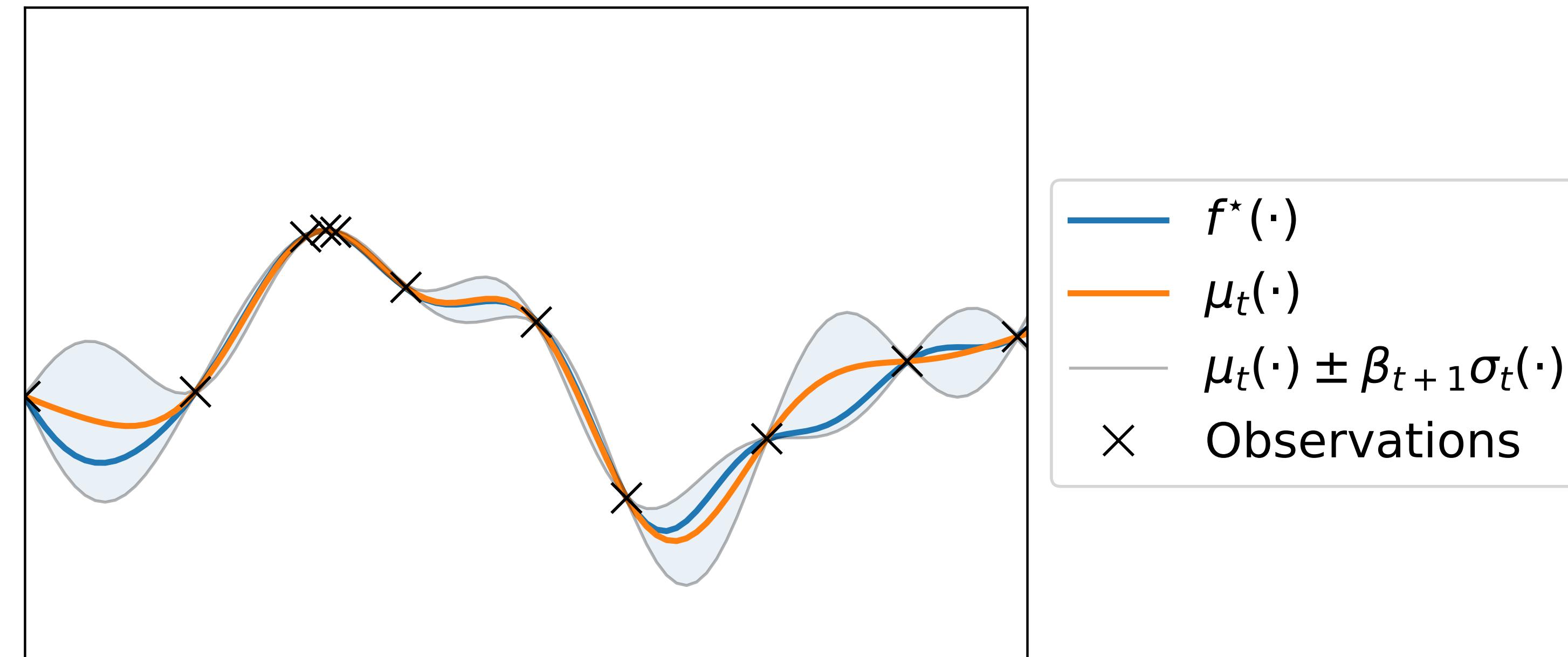
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enlargement

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- Enlargement is due to model misspecification: For all x, t it holds

$$|\mu_t(\cdot) - \mu_t^*(\cdot)| \leq \frac{\epsilon \sqrt{t}}{\sqrt{\lambda}} \sigma_t(\cdot)$$

- ▶ $\mu_t(\cdot)$: Hypothetical mean estimator that corresponds to the (noisy) observations of the best in-class function, i.e., $f \in \arg \min_{f \in \mathcal{F}_k(\mathcal{D}; B)} \|f - f^*\|_\infty$
- ▶ Intuition: Correlations among observations captured in the model increase the bias

Enlarged confidence UCB algorithm

- Recall assumptions:
 - ▶ Learner's hypothesis class $\mathcal{F}_k(\mathcal{D}; B)$; True reward function satisfies $\min_{f \in \mathcal{F}_k(\mathcal{D}; B)} \|f - f^*\|_\infty \leq \epsilon$

Theorem: EC-UCB achieves the following regret bound w.p. $1 - \delta$

$$R_T = \tilde{O} \left(\underbrace{(B + \sqrt{\ln(1/\delta)})\sqrt{\gamma_T T} + \gamma_T \sqrt{T}}_{\text{standard regret}} + T\epsilon\sqrt{\gamma_T} \right).$$

due to misspecification
(unavoidable for any algorithm)

- ▶ T : time horizon / number of samples
- ▶ ϵ : misspecification parameter
- ▶ γ_T : kernel dependent mutual information quantity (measure of function class complexity)

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- EC-UCB drawback: Requires knowing ϵ !!

Recall: $x_t = \arg \max_{x \in D} \mu_t^*(x) + \left(\beta_{t+1} + \frac{\epsilon\sqrt{t}}{\sqrt{\lambda}} \right) \sigma_t(x)$

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- Key idea: Use the uncertainty estimates to explore among “high-rewarding” actions
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- Note: \mathcal{D}_e might not contain the global maximizer since the confidence bounds are invalid
- Phased elimination in the misspecified linear bandit (Lattimore et al. '20)

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 - Recovers the linear misspecified setting and results (Lattimore et al.'20) when used with the linear kernel
 - Contemporary work of Camilleri et al.'21 obtains similar results via a very different algorithm
 - Misspecified kernelized contextual setting (in the paper)
 - ▶ Based on the regret bound balancing strategy of Pacchiano et al.'20

Summary

Goal:

- Protect against model misspecification
 - ▶ Bandits (Ghosh et al.'17, Zanette et al.'20, Lattimore et al.'20, Neu et al.'20, Foster et al.'20), Contextual bandits (Foster et al.'20, Pacchiano et al.'20), RL (Jin et al.'20, Du et al.'19)
- Strong model assumptions are restrictive in practice

Our contributions:

- Complete treatment of the misspecified GP bandit optimization problem
- Practical algorithms inspired by the classical Bayesian optimization and ED acquisition functions
- Theoretical regret bounds in multiple settings

Paper ID: 28196